

THE TIME OF BOOTSTRAP PERCOLATION WITH DENSE INITIAL SETS FOR ALL THRESHOLDS

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ABSTRACT. We study the percolation time of the r -neighbour bootstrap percolation model on the discrete torus $(\mathbb{Z}/n\mathbb{Z})^d$. For t at most a polylog function of n and initial infection probabilities within certain ranges depending on t , we prove that the percolation time of a random subset of the torus is exactly equal to t with high probability as n tends to infinity. This extends recent work of Bollobás, Holmgren, Smith and Uzzell, who proved similar results in the case $r = d$.

1. INTRODUCTION

Under r -neighbour bootstrap percolation on a graph G , some of the vertices of G are initially infected, and at each step, infected vertices stay infected, and uninfected vertices become infected if they have at least r infected neighbours. Making this formal, there is a set $A = A_0 \subset V(G)$, and for $t \geq 0$,

$$A_{t+1} = A_t \cup \{v : |\Gamma(v) \cap A_t| \geq r\}.$$

where $\Gamma(V)$ denotes the set of neighbours of v in the graph G . We write $[A] := \bigcup_{t=0}^{\infty} A_t$ for the *closure* of A , and say that A *percolates* G (or that *percolation occurs*) if eventually every vertex of G becomes infected; that is, if $[A] = V(G)$. The set A is *closed* if $[A] = A$.

Bootstrap percolation was introduced by Chalupa, Leath and Reich [12] as a model for certain interacting particle systems in physics. Since then it has found applications in crack formation, clustering phenomena, dynamics of glasses [19], sandpiles [17], the Ising model for ferromagnetism [25], jamming [21], and many other areas of statistical mechanics and physics, as well as in neural networks [29, 2], computer science [14, 18], and sociology [20, 30].

There are two broad classes of questions one can ask about bootstrap percolation. The first, and the most extensively studied, is what happens when the initial configuration A_0 is chosen randomly? Fix a probability p and let A_0 be a random subset of $V(G)$ in which vertices are included independently with probability p . One would like to know how likely percolation is to occur, and if it does occur, how long it takes.

The answer to the first of these questions is now well understood: on the lattice graph $[n]^d$, in which d is fixed and n tends to infinity, the probability of percolation under the r -neighbour model displays a sharp threshold between no percolation with high probability and percolation with high probability, meaning that there exists $p_c = p_c(n, d, r)$ such that for all $\epsilon > 0$, if $p \geq (1 + \epsilon)p_c$ then there is percolation with high probability, while if $p \leq (1 - \epsilon)p_c$ then there is no percolation with high probability. The existence of

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thresholds in a certain weaker sense was proved in papers by Aizenman, Lebowitz, Cerf, Cirillo and Manzo [1, 10, 11], and in the strong sense just described in papers by Holroyd, Balogh, Bollobás, Duminil-Copin and Morris [22, 5, 4]. Sharp thresholds have also been proved for the hypercube (Balogh and Bollobás [3], and Balogh, Bollobás and Morris [6]) and for several other bootstrap models on \mathbb{Z}^d (Duminil-Copin and Holroyd [15], and Duminil-Copin and van Enter [16]).

If p is large enough for percolation to occur with high probability, one would like to know how long percolation takes. In other words, what can one say about the random variable

$$T = \min\{t : A_t = V(G)\}?$$

(If percolation does not occur then we define $T = \infty$.) The question has been answered by Janson, Łuczak, Turova and Vallier [23] in the case in which G is the Erdős-Rényi random graph $G(n, p)$. In a recent paper of Bollobás, Holmgren, Smith and Uzzell [9] the question is studied on the discrete torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ under the d -neighbour model for small values of t . For $t = t(n)$ up to $o(\log n / \log \log n)$ (including constant t) it is determined that for certain ranges of the initial probability $p = p(n)$ one can say that $T = t$ with high probability, while for other ranges one obtains $T \in \{t, t+1\}$ with high probability. In this paper we extend these results to the r -neighbour model for all $2 \leq r \leq 2d$.

We write q_n for $1 - p_n$ and q for $1 - p$. Let

$$m_{d,r}(t) = \sum_{i_0=0}^t \sum_{i_1=0}^{i_0} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}. \quad (1)$$

This iterated sum, which will appear frequently throughout the paper, is the size of a certain set that is naturally associated with the bootstrap process. We shall be more precise about what we mean by this later in the introduction. The following two theorems are our main results.

Theorem 1. *Let $d \geq r \geq 2$, let $t = t(n) = o((\log n / \log \log n)^{1/(d-r+1)})$, let $(p_n)_{n=1}^\infty$ be a sequence of probabilities, and let $\omega(n) \rightarrow \infty$. Under the standard r -neighbour rule on \mathbb{T}_n^d ,*

- (i) *if $q_n \leq (n^{-d}/\omega(n))^{1/m_{d,r}(t)}$, then $\mathbb{P}_{p_n}(T \leq t) \rightarrow 1$ as $n \rightarrow \infty$;*
- (ii) *if $q_n \geq (n^{-d}\omega(n))^{1/m_{d,r}(t)}$, then $\mathbb{P}_{p_n}(T \leq t) \rightarrow 0$ as $n \rightarrow \infty$.*

In addition to Theorem 1, we prove that if the sequence $(q_n)_{n=1}^\infty$ satisfies certain bounds, then the percolation time T is determined exactly, or that it takes one of two values, in each case with high probability as n tends to infinity.

Theorem 2. *Let $d \geq r \geq 2$, let $t = o((\log n / \log \log n)^{1/(d-r+1)})$, and let $(p_n)_{n=1}^\infty$ be a sequence of probabilities.*

- (i) *Suppose there exists $\omega(n) \rightarrow \infty$ such that*

$$(n^{-d}\omega(n))^{1/m_{d,r}(t-1)} \leq q_n \leq (n^{-d}/\omega(n))^{1/m_{d,r}(t)}.$$

Then $T = t$ with high probability.

- (ii) *Suppose instead that*

$$(n^{-d}/\omega(n))^{1/m_{d,r}(t)} \leq q_n \leq (n^{-d}\omega(n))^{1/m_{d,r}(t)}$$

for all functions $\omega(n) \rightarrow \infty$. Then $T \in \{t, t+1\}$ with high probability. If, moreover, there exists a constant $c > 0$ such that $q_n^{m_{d,r}(t)} n^d \rightarrow c$ as $n \rightarrow \infty$, then

$$\mathbb{P}_{p_n}(T = t) \sim 1 - \mathbb{P}_{p_n}(T = t+1) \sim \exp(-g_{d,r}c),$$

where

$$g_{d,r} = \binom{d}{d-r+1} 2^{r-1} d^{2(d-r+1)}.$$

When $r > d$, the situation is very different. For r in this range, which is called the *subcritical* range, there exist cofinite sets in \mathbb{Z}^d which are closed under the r -neighbour model. This greatly simplifies the analysis of the model, and we shall see in Theorem 19 that it reduces the range of possible percolation times to the finite set $\{0, 1, \dots, d, \infty\}$ with high probability.

The second broad class of questions one can ask about bootstrap percolation is the class of extremal questions: for example, what is the minimum or maximum size of A_0 such that a certain property holds, or what is the minimum or maximum time that percolation can take, possibly given certain properties of A_0 ? The first significant theorems in extremal bootstrap percolation are due to Morris [24] and Riedl [27], who studied the sizes of minimal percolating sets on the square grid and the hypercube $\{0, 1\}^d$ respectively. Later, Benevides and Przykucki [8] determined the maximal percolating time on the square grid, and Przykucki [26] did the same for the hypercube.

In [9], Theorem 1 is proved in the case $r = d$. The central part of the proof consists of a series of three extremal results. The first of these supposes that a given site $x \in \mathbb{T}_n^d$ is uninfected at time t and asks what is the maximum possible size of the set of initially infected sites A_0 ? A preliminary observation is that the states of sites at ℓ_1 distance greater than t from x cannot affect whether or not x is infected at time t , so we may restrict our attention to the ℓ_1 ball

$$B_t(x) = \{y \in \mathbb{T}_n^d : \|x - y\| \leq t\}.$$

Later we shall need the ℓ_1 sphere

$$S_t(x) = \{y \in \mathbb{T}_n^d : \|y - x\| = t\},$$

and we write B_t for $B_t(0)$ and S_t for $S_t(0)$. The question now is to determine the quantity

$$\text{ex}_{d,r}(t) := \min\{|B_t \setminus A_0| : 0 \notin A_t\}. \quad (2)$$

If a set $P \subset V(G)$ of vertices has the property that, provided no element of P is in A_0 , then no matter what the initial states of the other vertices in $V(G)$ are it can never be the case that $x \in A_t$, then we say that P *protects* x (the time t is implicit). Given $x \in \mathbb{T}_n^d$, let x_i be the i th coordinate of x with respect to the standard basis vectors e_1, \dots, e_d in \mathbb{R}^d , so $x = (x_1, \dots, x_d)$. A natural example of a set of sites that protects 0 under the d -neighbour process is the column

$$\{x \in B_t : x_2, \dots, x_d \in \{0, 1\}\}.$$

The first extremal theorem in [9] says that this is best possible: $\text{ex}_{d,d}(t)$ is at least the size of such a column for every t . The second extremal theorem in [9] states that these columns are essentially the only sets achieving the minimum ('essentially' meaning that there do exist other extremal sets, but there are a fixed number of them, and they only differ from a column in the positions of at most two sites).

Suppose instead that the number of initially infected sites is not the minimal number, but is close to the minimum. What can we say about the positions of these sites now? The third extremal theorem in [9] is a stability theorem which says that under these conditions the sites must look a lot like a column, in a certain specific sense.

These three extremal results are not only crucial to proving the main theorem in [9] (which is the case $r = d$ of Theorem 1 in this paper), but are interesting results in their

own right. In this paper we prove generalizations of these results to arbitrary thresholds r , and once again, these results form the backbone of the proof of Theorem 1.

For general r , a natural set that protects the origin is

$$P = \{x \in B_t : x_{d-r+2}, \dots, x_d \in \{0, 1\}\}, \quad (3)$$

which we can think of as the intersection of $B_t(0)$ with a (disjoint) union of 2^{r-1} translates of the $(d-r+1)$ -dimensional ‘subspace’ $\{x : x_{d-r+2} = \dots = x_d = 0\}$ in \mathbb{Z}^d , or informally as a $(d-r+1)$ -dimensional set in \mathbb{Z}^d with ‘thickness’ 2. (Of course \mathbb{Z}^d is not a vector space, so it does not make sense to talk about subspaces, but we shall often do so, unambiguously, always meaning the intersection of \mathbb{Z}^d with the corresponding subspace of \mathbb{R}^d . The purpose of this slight abuse of nomenclature is to make clear the graph structure of the object under consideration, and in particular the degrees of the sites.) With $m_{d,r}(t)$ as in (1), we prove that

$$\text{ex}_{d,r}(t) = |P| = m_{d,r}(t). \quad (4)$$

We also prove that sets of the form of P are essentially the only extremal sets, and we prove a stability theorem for near-minimal sets.

The proofs of the first and third extremal results use the corresponding results from [9] as the base case for an induction argument on the difference between the dimension d and threshold r . The proof of the second extremal result is a generalization of the corresponding proof from [9], but here the argument is more streamlined.

There are also versions of our results for the *modified* r -neighbour bootstrap percolation model. In this model, on the graph \mathbb{T}_n^d , there is again an initial set A_0 of infected sites, and for $t \geq 0$ we set

$$A_{t+1} = A_t \cup \{v : |\{v + e_i, v - e_i\} \cap A_t| \geq 1 \text{ for at least } r \text{ distinct choices of } i \in [d]\}.$$

With some slight simplifications, our arguments can be used to prove the following result.

Theorem 3. *Under the modified r -neighbour rule on \mathbb{T}_n^d , Theorems 1 and 2 hold, with $m_{d,r}(t)$ replaced by $m'_{d,r}(t)$ and $g_{d,r}$ replaced by $g'_{d,r}$, where*

$$m'_{d,r}(t) = \sum_{i_0=0}^{d-r+1} \binom{d-r+1}{i_0} \sum_{i_1=0}^{t-i_0} \sum_{i_2=0}^{i_1} \dots \sum_{i_{d-r+1}=0}^{i_{d-r}} 1$$

is the volume of a $(d-r+1)$ -dimensional ℓ_1 ball of radius t , and

$$g'_{d,r} = \binom{d}{d-r+1}.$$

The rest of this paper is organized as follows. In Section 2 we prove two basic results about binomial coefficients, which are needed in the proofs of our main extremal results. In Sections 3 and 4 we study minimal and near-minimal protecting sets respectively, and prove the three extremal theorems. We bring these results together in Section 5, and with the help of some standard probabilistic tools, use them to prove Theorems 1 and 2. Finally, in Section 6, we prove corresponding results for subcritical models.

2. COMBINATORIAL PRELIMINARIES

The purpose of this section is to prove two easy identities concerning sums of binomial coefficients; we shall use these repeatedly throughout the next two sections.

Lemma 4. *Let $d \geq r \geq 2$, let $f \geq 0$, and let $k \geq 0$. Then*

$$\begin{aligned} \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} \\ = 2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + \sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}}. \end{aligned} \quad (5)$$

In Section 3 we show that the left-hand side of (5) is the volume of the surface of an extremal set of radius k , as in (3). The lemma can be thought of as saying that this volume is equal to the volume of that part of the surface that lies in a co-dimension 1 subspace (the second term on the right-hand side of (5)) plus the volume of the surface that lies in each hyperplane parallel to (but distinct from) the subspace.

Proof. We shall prove the identity by induction on $d-r$. When $d-r=0$, (5) is equivalent to

$$\sum_{i_1=0}^k \binom{f}{i_1} = 2 \sum_{i_1=0}^{k-1} \binom{f-1}{i_1} + \binom{f-1}{k}.$$

Rewriting the right-hand side as

$$\binom{f-1}{0} + \left(\binom{f-1}{0} + \binom{f-1}{1} \right) + \cdots + \left(\binom{f-1}{k-1} + \binom{f-1}{k} \right),$$

we see that the identity holds.

Suppose the lemma holds for $d-r-1$. After re-indexing the second expression, the right-hand side of (5) is equal to

$$2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}}. \quad (6)$$

We can rewrite the first expression here as

$$\begin{aligned} 2 \sum_{i_1=0}^{k-1} \left(\sum_{i_2=0}^{i_1-1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + \sum_{i_3=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} \right) \\ = \sum_{i_1=0}^{k-1} 2 \sum_{i_2=0}^{i_1-1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + 2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}}, \end{aligned} \quad (7)$$

where in the second line we have just moved the factor of 2 inside one sum and re-indexed the second sum. We can also rewrite the second expression in (6) as

$$\sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}} + \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}}. \quad (8)$$

Summing (7) and (8) we obtain that (6) is equal to

$$\begin{aligned} \sum_{i_1=0}^{k-1} \left(2 \sum_{i_2=0}^{i_1-1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}} \right) \\ + \left(2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}} + \sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f-1}{i_{d-r}} \right). \end{aligned}$$

Applying induction twice, once to each of the expressions inside the two sets of large brackets, we find this is equal to

$$\sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} + \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r}=0}^{i_{d-r-1}} \binom{f}{i_{d-r}},$$

which, upon re-indexing the second expression one last time and combining the sums, proves the lemma. \square

The next lemma is an iterated version of the previous one, and has a similar iterated interpretation.

Lemma 5. *Let $d > r \geq 2$, let $0 \leq f \leq d$, and let $k \geq 0$. Then*

$$\begin{aligned} \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} &= 2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} \\ &+ 2 \sum_{i_2=0}^{k-1} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-2}{i_{d-r+1}} + \cdots \\ &+ 2 \sum_{i_{d-r+1}=0}^{k-1} \binom{f-d+r-1}{i_{d-r+1}} + \binom{f-d+r-1}{k}. \quad (9) \end{aligned}$$

Proof. Again, we shall prove the identity by induction on $d-r$. When $d-r=1$, the claim is precisely the same as Lemma 4.

Suppose the lemma holds for $d-r-1$. It follows that the right-hand side of (9) is equal to

$$2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}} + \sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f-1}{i_{d-r+1}}.$$

This is equal to the left-hand side of (9) by Lemma 4. \square

3. MINIMAL CONFIGURATIONS

In this section we prove two extremal theorems about sets of uninfected sites with certain properties. The first theorem determines the extremal number of sites defined in (2) and shows that it is equal to the size of the natural example of a protecting set defined in (3). The second says that sets like the one in (3) are essentially the only extremal sets.

We need quite a lot of notation before we can state the results of this section in the level of generality that we need later in the paper.

First, define a partial order on sites in \mathbb{Z}^d as follows. For $x, y \in \mathbb{Z}^d$, we say that $x \leq y$, or that $y \geq x$, or that y is *above* x , if $y_i \geq x_i$ for each i such that $x_i > 0$, and $y_i \leq x_i$ for each i such that $x_i < 0$.

For a fixed t , a site $x \in \mathbb{T}_n^d$ is said to be *protected* if $x \notin A_t$. Saying that a site is protected is therefore a stronger statement than saying that it is initially uninfected.

Two neighbours y_1 and y_2 of a site x are said to be *opposing* if there exists i such that $y_1 = x + e_i$ and $y_2 = x - e_i$, or vice-versa.

The next few definitions allow us to be more specific about where we are looking for sites protecting a given site. The extra control we gain over the positions of these sites is not needed in this section, but it will be necessary for the stability result in the next section.

A function $C : [d] \rightarrow \{-1, 0, 1, *\}$ is called a *compatibility function*. A site y is C -compatible with a site x if the following three conditions hold:

- (i) $y_i - x_i \geq 0$ if $C(i) = 1$;
- (ii) $y_i - x_i \leq 0$ if $C(i) = -1$;
- (iii) $y_i = x_i$ if $C(i) = 0$.

If $C(i) = *$ then there is no restriction on y_i . Given a compatibility function C , we write $\mathcal{P}(C)$ for the set $\{i : C(i) = 1\}$ of *positive coordinates*, $\mathcal{N}(C)$ for the set $\{i : C(i) = -1\}$ of *negative coordinates*, $\mathcal{Z}(C)$ for the set $\{i : C(i) = 0\}$ of *fixed coordinates*, and $\mathcal{F}(C)$ for the set $\{i : C(i) = *\}$ of *free coordinates*. For a site x , an integer $k \geq 0$ and a compatibility function C , let $P_k^C(x)$ denote the set of all protected sites in $S_k(x)$ which are C -compatible with x . The i -restriction of a compatibility function C is the compatibility function C' which satisfies $C'(i) = 0$ and $C'(j) = C(j)$ for all $j \neq i$.

The following lemma for subcritical $(d+1)$ -neighbour bootstrap percolation is proved in [9], although it is not explicitly stated, and it is also a special case of Lemma 17 in this paper.

Lemma 6. *Let $k \in \mathbb{N}$ and let $d \geq 2$. Suppose that $x \in B_k$ is protected under $(d+1)$ -neighbour bootstrap percolation. Let C be a compatibility function with no fixed coordinates and let $f = |\mathcal{F}(C)|$ be the number of free coordinates of C . Then*

$$|P_k^C(x)| \geq \binom{f}{k}. \quad \square$$

The next lemma will be the key lemma in the proof of the first extremal theorem.

Lemma 7. *Let $k \in \mathbb{N}$ and let $2 \leq r \leq d$. Suppose that $x \in B_k$ is protected under r -neighbour bootstrap percolation. Let C be a compatibility function with no fixed coordinates and let $f = |\mathcal{F}(C)|$ be the number of free coordinates of C . Then*

$$|P_k^C(x)| \geq \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}}.$$

In the applications of Lemma 7 in this section we shall always take $\mathcal{P}(C) = \{i : x_i > 0\}$, $\mathcal{N}(C) = \{i : x_i < 0\}$, $\mathcal{F}(C) = \{i : x_i = 0\}$, and $\mathcal{Z}(C) = \emptyset$. It may be helpful to have this in mind during the proof.

Here is an outline of the proof. We assume that $\mathcal{P}(C) = [d] \setminus \mathcal{F}(C) = [d - f]$. The argument runs by induction on the sum $(d - r) + k + f$. Base cases and the case $r = d$ are proved in [9]. There are two possibilities for the induction step. First, suppose that x has at least $d - r + 1$ pairs of opposing protected neighbours. We may assume that they are $x \pm e_{d-f+1}, \dots, x \pm e_{2d-f-r+1}$. Now divide up the sites above x (thinking of C as in the previous paragraph) in B_k as follows. First, take the sites above either $x + e_{d-f+1}$ or $x - e_{d-f+1}$. These sets look like half spaces, in the sense that the only restriction is on the sign of the $(d - f + 1)$ th coordinate. Next, take the sites not in either of those

sets (so their $(d - f + 1)$ th coordinate is zero), but which are above either $x + e_{d-f+2}$ or $x - e_{d-f+2}$. These sets look like half-spaces inside hyperplanes, because of the restrictions that the $(d - f + 1)$ th coordinate is zero and the sign of the next coordinate is fixed. We continue, looking at smaller sets each time. Inside each of these sets, which are disjoint, we apply the induction hypothesis separately, and then we use Lemma 5 to show that we have found the right number of protected sites. This completes the case when x has lots of opposing protected neighbours. Now suppose x has at most $d - r$ pairs of opposing protected neighbours. In this case, which is the easier of the two, x must have a protected neighbour in a direction among the first $d - f$ coordinates, which we may assume is $x + e_1$. By induction, there are lots of protected sites above $x + e_1$, and also by induction we can find lots of protected sites above x but with first coordinate fixed. It then turns out that the total number of protected sites we get is the right number.

Proof of Lemma 7. Without loss of generality, let $\mathcal{P}(C) = [d] \setminus \mathcal{F}(C) = [d - f]$. Let $x = (x_1, \dots, x_d)$ and suppose that $x_i \geq 0$ for all $i \in [d]$.

The proof is by induction on $q := (d - r) + f + k$. If $q = 0$ then we must have $k = 0$ and the claim is simply that x itself is protected, which is trivial. The lemma is proved for all k and f when $r = d$ in [9].

As in the proof for d -neighbour bootstrap percolation, there are two cases to consider (not $d - r + 2$ cases, as one might suppose).

Case 1: x has at most $d - r$ pairs of opposing C -compatible protected neighbours. Since x has at least $2d - r + 1$ protected neighbours, of which at most $d - f$ are not C -compatible, it must have at least $d + f - r + 1$ C -compatible protected neighbours. Now, C has f free coordinates, and x has at most $d - r$ pairs of opposing C -compatible protected neighbours, so it has at most $d + f - r$ protected neighbours of the form $x + e_i$ or $x - e_i$, where $i \in \{d - f + 1, \dots, d\}$. It follows that x has at least one protected neighbour of the form $x + e_i$, for some $i \in [d - f]$. Let us assume that $x' := x + e_1$ is protected. Observe that $P_{k-1}^C(x') \subset P_k^C(x)$, since x' differs from x only in one of its positive coordinates. By induction,

$$|P_{k-1}^C(x')| \geq \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}}. \quad (10)$$

Let C' be the 1-restriction of C . The remaining sites that we need to complete this case of the proof will be in $P_k^{C'}(x)$. Observe that

$$P_{k-1}^C(x') \cap P_k^{C'}(x) = \emptyset, \quad (11)$$

because sites in $P_{k-1}^C(x')$ have first coordinate at least $x_1 + 1$ and sites in $P_k^{C'}(x)$ have first coordinate exactly x_1 . Let U be the co-dimension 1 subspace given by

$$U = \{y \in \mathbb{Z}^d : y_1 = x_1\}.$$

The set of sites which are C' -compatible with x in \mathbb{Z}^d is the same as the set of sites which are C -compatible with x in U . Thus, by induction there are at least

$$\sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} \quad (12)$$

C -compatible protected sites at distance k from x all contained in U . By (10), (11) and (12), the total number number of C -compatible protected sites at distance k from x is

$$\begin{aligned} |P_k^C(x)| &\geq \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} + \sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}} \\ &= \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{f}{i_{d-r+1}}, \end{aligned}$$

as claimed.

Case 2: x has at least $d - r + 1$ pairs of opposing C -compatible protected neighbours. Without loss of generality the opposing protected neighbours are $x \pm e_{d-f+1}, \dots, x \pm e_{2d-f-r+1}$. Let $C_1 = C$ and for $i = 1, \dots, d - r + 1$ let C_{i+1} be the $(d - f + i)$ -restriction of C_i . Also, for $i = 1, \dots, d - r + 1$, let C_i^+ be the compatibility function satisfying $C_i^+(d - f + i) = 1$ and $C_i^+(j) = C_i(j)$ for $j \neq i$, and let C_i^- be the compatibility function satisfying $C_i^-(d - f + i) = -1$ and $C_i^-(j) = C_i(j)$ for $j \neq i$. The key observation here is that the sets $P_{k-1}^{C_1^+}(x + e_{d-f+1})$, $P_{k-1}^{C_1^-}(x - e_{d-f+1})$, \dots , $P_{k-1}^{C_{d-r+1}^+}(x + e_{2d-f-r+1})$, $P_{k-1}^{C_{d-r+1}^-}(x - e_{2d-f-r+1})$, and $P_{k-1}^{C_{d-r+2}}(x)$ are all pairwise disjoint, so we can obtain a bound on $P_k^C(x)$ by bounding the sizes of each of these sets individually.

The compatibility function C_i has exactly $i - 1$ fixed coordinates. In fact, all sites that are C_i -compatible with x lie inside the $(d - i + 1)$ -dimensional affine subspace

$$U_i = \{y \in \mathbb{Z}^d : y_{d-f+1} = x_{d-f+1}, \dots, y_{d-f+i-1} = x_{d-f+i-1}\}.$$

(We define $U_1 = \mathbb{Z}^d$.) Thus, when we are looking for C_i -compatible protected sites, we are really looking for protected sites inside a $(d - i + 1)$ -dimensional space. The function C_i has $f - i + 1$ free coordinates, and C_i^+ and C_i^- each have $f - i$ free coordinates. By induction, for each $i = 1, \dots, d - r + 1$ we have

$$|P_{k-1}^{C_i^+}(x + e_{d-f+i})| \geq \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-i-r+2}=0}^{i_{d-i-r+1}} \binom{f-i}{i_{d-i-r+2}}, \quad (13)$$

and a similar inequality holds for $|P_{k-1}^{C_i^-}(x - e_{d-f+i})|$. Furthermore, C_{d-r+2} has $f - d + r - 1$ free coordinates, and sites which are C_{d-r+2} -compatible with x all lie in the $(r - 1)$ -dimensional affine subspace U_{d-r+2} . By Lemma 6,

$$|P_{k-1}^{C_{d-r+2}}(x)| \geq \binom{f-d+r-1}{k}. \quad (14)$$

Summing (13) over $i = 1, \dots, d - r + 1$ and each choice of $+$ or $-$, and adding (14) to the sum, we obtain precisely the right-hand side of the identity (9). Lemma 5 then completes this case of the proof. \square

Corollary 8. *Let $t \in \mathbb{N}$ and let $2 \leq r \leq d$. Suppose that the origin is protected under r -neighbour bootstrap percolation. Then for $k = 0, \dots, t$,*

$$|P(S_k)| \geq \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}.$$

Consequently,

$$|P(B_t)| \geq m_{d,r}(t) = \sum_{i_0=0}^t \sum_{i_1=0}^{i_0} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}. \quad \square$$

Corollary 8 is not quite what we have so far described as the first extremal result. It determines the exact value of $\text{ex}_{d,r}(t)$, but we do not yet know that this is the size of the natural candidate for an extremal set, defined in (3). Before we verify this claim, we define these sets in full generality.

A subset K of $B_t(x)$ is (d, r) -canonical if there is a subset I of $[d]$ of size $r - 1$ and $\epsilon_i \in \{-1, 1\}$ for each $i \in I$ such that

$$K = \{y \in B_t(x) : y_i - x_i \in \{0, \epsilon_i\} \text{ for all } i \in I\}.$$

The elements of the set K are called *canonical sites*. The ball $B_t(x)$ will often be implicit, but where needed we call the parameter t the *radius* of K . The set I is the *orientation* of K and its complement $[d] \setminus I$ is the *alignment* of K . A coordinate $i \in I$ is an *orientation coordinate* and a coordinate $j \in [d] \setminus I$ is an *alignment coordinate*.

Given x , and I and ϵ_i as above, let

$$E_j^+ = \{x + te_j\} \cup \{x + (t-1)e_j - \epsilon_i e_i : i \in I\}$$

for $j \in [d] \setminus I$, and similarly let

$$E_j^- = \{x - te_j\} \cup \{x - (t-1)e_j - \epsilon_i e_i : i \in I\}.$$

for $j \in [d] \setminus I$. Let E be any set consisting of exactly one site from each of the E_j^+ and each of the E_j^- , so $|E| = 2(d - r + 1)$. A subset K' of $B_t(x)$ is (d, r) -semi-canonical if there is a (d, r) -canonical set K and a choice of E (with the alignment and orientation given by K) such that

$$K' = (K \setminus \{x + te_j, x - te_j : j \in [d] \setminus I\}) \cup E.$$

We call the sites in E the *extreme sites* of the (d, r) -semi-canonical set K' .

Lemma 9. *Let $t \in \mathbb{N}$ and let $2 \leq r \leq d$. Suppose $K \subset B_t$ is (d, r) -canonical. Then $|K| = m_{d,r}(t)$.*

Note that a (d, r) -semi-canonical set has the same size as a (d, r) -canonical set (with the same radius), so Lemma 9 also applies to these sets.

Proof. The induction is on d . Let K_k be the intersection of K with S_k . We shall prove that

$$|K_k| = \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}$$

for $k = 0, \dots, t$, which will prove the lemma.

Let the alignment set be $[d - r + 1]$ and the orientation set $[d] \setminus [d - r + 1]$. When $d = r$ there is exactly one alignment coordinate, so

$$|K_k| = \binom{r-1}{k} + 2 \sum_{i=0}^{k-1} \binom{r-1}{i} = \sum_{i=0}^k \binom{r}{i},$$

the second equality following from Lemma 4.

Suppose the lemma holds for $d - 1$. The set of sites in K_k with first coordinate zero is a $(d - 1, r)$ -canonical set, so the number of such sites is

$$\sum_{i_2=0}^k \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}$$

by the induction hypothesis. Now fix i such that $0 \leq i \leq k - 1$; negative i are treated similarly by symmetry. The set of sites in K_k with first coordinate $k - i$ is a $(d - 1, r)$ -canonical set of radius i in the affine subspace $\{x : x_1 = k - i\}$, so by induction the number of such sites is

$$\sum_{i_2=0}^i \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}.$$

Summing over i and doubling to take account of the choice of sign of x_1 , the number of sites in K_k with first coordinate non-zero is

$$2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}.$$

The result now follows from Lemma 4. \square

Given that the origin is protected, we say that the set $P(B_t)$ is *minimal* if it has size $m_{d,r}(t)$. For any minimal set $P(B_t)$, we define $l_{d,r}(t)$ to be the number of sites in $P(S_t)$, which by Lemmas 7 and 9 is equal to $m_{d,r+1}(t)$ and to the size of the intersection of a (d, r) -canonical set with S_t . Thus,

$$l_{d,r}(t) = \sum_{i_1=0}^t \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}.$$

Usually d and r will be clear from the context, so we shall write $l(t)$ and $m(t)$ for $l_{d,r}(t)$ and $m_{d,r}(t)$ respectively. Given that the origin is protected, the set $P(S_t)$ is *minimal* if it has size $l_{d,r}(t)$.

The following result, which is a combination of Corollary 8 and Lemma 9, is the full version of the first extremal theorem.

Theorem 10. *Let $t \in \mathbb{N}$ and let $2 \leq r \leq d$. Suppose that the origin is protected under r -neighbour bootstrap percolation. Then $|P(S_k)| \geq l_{d,r}(k)$ for $k = 0, \dots, t$, and*

$$|P(B_t)| \geq |K| = m_{d,r}(t),$$

where K is any (d, r) -canonical set.

The next theorem states that all minimal sets protecting the origin are semi-canonical. It is the second extremal theorem.

Theorem 11. *Let $t \geq 2$ and let $2 \leq r \leq d$. Suppose that the origin is protected under r -neighbour bootstrap percolation and that $P(B_t)$ is minimal. Then $P(B_t)$ is (d, r) -semi-canonical.*

Proof. Throughout the proof we write P_k for $P(S_k)$ for each $k \in [t]$. Since the origin is protected, it must have at least $2d - r + 1$ protected neighbours. Therefore, by the pigeonhole principle, it must have at least $d - r + 1$ pairs of opposing protected neighbours. Suppose it has at least $d - r + 2$ pairs of opposing protected neighbours. Let R_2 be the

set of sites in S_2 which have degree at least 2 (and hence exactly 2) in P_1 . R_2 is precisely the set of all $x + y$ such that x and y belong to P_1 and $x + y \neq 0$. Thus,

$$|R_2| \leq \binom{2d-r+1}{2} - (d-r+2). \quad (15)$$

It is easy to verify (for example, by using Lemma 4 and induction on $d-r$) that

$$l_{d,r}(2) = \sum_{i_1=0}^2 \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}} = \binom{2d-r+1}{2} + (d-r+1),$$

which means that the right-hand side of (15) is exactly $2(d-r+1) + 1$ less than $l_{d,r}(2)$. Therefore,

$$\sum_{x \in P_2} d_{P_1}(x) \leq \sum_{x \in R_2} d_{P_1}(x) + 2(d-r+1) + 1 = 2|R_2| + 2(d-r+1) + 1. \quad (16)$$

Every $x \in P_1$ has at least $2d-r+1$ protected neighbours, of which one is the origin and the rest are in S_2 . The total out degree of P_1 is thus

$$\sum_{x \in P_1} d_{P_2}(x) = (2d-r)|P_1| = (2d-r)(2d-r+1). \quad (17)$$

The combination of (15), (16) and (17) is a contradiction. We conclude that the origin has exactly $d-r+1$ pairs of opposing protected neighbours. Without loss of generality, let

$$P_1 = \{e_1, \dots, e_d, -e_1, \dots, -e_{d-r+1}\}.$$

This is the base case of our induction. From now on we assume that P_{k-1} is canonical and our aim is to prove that P_k must be canonical. Later we show that if P_{k-1} is semi-canonical but not canonical then we cannot extend the set of protected sites to the k th sphere, which will be a contradiction.

For the induction step, observe that on the one hand, every site in P_{k-1} must have at least a certain fixed number of edges into P_k to ensure that it is protected, while on the other hand, we are only allowed a certain fixed number of sites in P_k to achieve this, so we need to choose relatively few sites in S_k with relatively large degree into P_{k-1} . We shall show that the canonical vertices in S_k (or more precisely, those with degree at least 2 into P_{k-1}) are the only vertices in S_k with enough edges into P_{k-1} to achieve this aim. To prove this, let R_k be the set of sites in S_k with degree at least 2 in P_{k-1} . We know exactly what P_{k-1} is: it is the intersection of S_{k-1} with the (d, r) -canonical set having alignment coordinates $1, \dots, d-r+1$. Therefore, we can also easily see exactly what R_k is: it is the intersection of S_k with the same canonical set, excluding the extreme vertices. It follows that $|R_k|$ is exactly $2(d-r+1)$ less than $l_{d,r}(k)$, by Lemma 9. The protected set P_k is minimal and therefore has size $l_{d,r}(k)$ by Theorem 10, so we must have

$$\sum_{x \in P_k} d_{P_{k-1}}(x) \leq \sum_{x \in R_k} d_{P_{k-1}}(x) + 2(d-r+1). \quad (18)$$

If there is equality in (18) then it must be the case that $R_k \subset P_k$, because R_k contains every site in S_k with degree at least 2 into P_{k-1} . Also, we have just observed that R_k consists of all non-extreme canonical sites in S_k . Together with the minimality of P_k and Theorem 10, this implies that

$$\sum_{x \in R_k} d_{P_{k-1}}(x) = \sum_{x \in P_{k-1}} d_{P_k}(x) - 2(d-r+1). \quad (19)$$

Thus we have equality in (18), and as noted above, this means that $R_k \subset P_k$. The only sites in P_{k-1} that do not have enough protected neighbours in P_k so far are the extreme sites, and they all need exactly one more protected neighbour. It follows that P_k must be (d, r) -semi-canonical.

Now suppose P_{k-1} is semi-canonical but not canonical, so we have $m \geq 1$ extreme sites x_1, \dots, x_m not of the form $\pm(k-1)e_i$ for any i , and remaining extreme sites $y_1, \dots, y_{2(d-r+1)-m}$. Let R'_k be the set of sites in S_k with degree at least 2 in P_{k-1} . Vertices in S_k cannot be adjacent to more than one of the x_i , and if they are adjacent to exactly one then have at least one strictly negative coordinate. Therefore the degree of x_i in R'_k is zero for $i = 1, \dots, m$ and the degree of y_i in R'_k is the same as in R_k , which is $2d - r - 1$. There were $2d - r - 1$ sites in R_k that were neighbours of a given extreme site and had degree exactly 2 in P_{k-1} , so we have $|R'_k| = |R_k| - m(2d - r - 1)$ and hence

$$|P_k| = |R'_k| + 2(d - r + 1) + m(2d - r - 1).$$

Therefore

$$\sum_{x \in P_k} d_{P_{k-1}}(x) \leq \sum_{x \in R'_k} d_{P_{k-1}}(x) + 2(d - r + 1) + m(2d - r - 1).$$

Let K_{k-1} be the set of canonical sites in S_{k-1} . Since we have lost $2m(2d - r - 1)$ edges in total between the two spheres by changing K_{k-1} to P_{k-1} and R_k to R'_k , we have

$$\sum_{x \in R'_k} d_{P_{k-1}}(x) \leq \sum_{x \in R_k} d_{K_{k-1}}(x) - 2m(2d - r - 1).$$

Combining the last two inequalities with (19) leads to a contradiction if $m \geq 1$ and $2d - r - 1 \geq 1$, which are both true. \square

4. NEAR-MINIMAL CONFIGURATIONS

We turn to the third of the three extremal theorems, which gives a rough description of near-minimal protecting sets. The theorem is as follows.

Theorem 12. *Let $2 \leq r \leq d$. There exist c_1 and c_2 depending only on d such that the following holds. Suppose there exists $k_1 \geq c_1$ such that $P(S_k)$ is minimal for all k in the range $k_1 \leq k \leq k_1 + c_2$. Let $t \geq k_1 + c_2$ and suppose that the origin is protected. Then $P(B_{k_1-c_1})$ is (d, r) -canonical.*

Thus, we are able to say the following. If the origin is protected and the protected sites in at least a fixed constant number of spheres are minimal, then the protected sites in all but a final (different) fixed constant number of layers form a canonical set. The corollary of this that we need in the proof of Theorem 1 is that if the origin is protected and there are $m(t) + a$ protected sites in B_t , then the number of possible configurations of the protected sites is $t^{O(a)}$. The trivial bound would be $t^{O(ta)}$.

The proof is by induction on $d - r$. We show that the intersections of $P(S_k)$ with each of a set of parallel hyperplanes are all minimal for a certain range of k . We can apply induction inside these hyperplanes, which are $(d - 1)$ -dimensional, to find $(d - 1, r)$ -canonical sets. We have to show that these canonical sets line up: their alignment and orientation coordinates must match. To do this, we repeat what we have done with an orthogonal subgroup of \mathbb{Z}^d , and using these sets we can show that if any of the alignments or orientations of the original sets did not agree then there would be too many protected sites.

Proof. The case $d = r$ is covered by Theorem 11 in [9], which gives constants $c_1 = d$ and $c_2 = 3d + 1$. Here we shall let $r \leq d - 1$ and proceed by induction on $d - r$, assuming that the result holds for $d - r - 1$ with constants c_1 and c_2 fixed throughout the proof.

In what follows, k will always be assumed to be in the range $k_1 \leq k \leq k_1 + c_2$. Let

$$U_i = \{x \in \mathbb{Z}^d : x_1 = i\}$$

be a hyperplane for each i . The origin must have at least $2d - r + 1 \geq d + 2$ protected neighbours, so we may assume that $\pm e_1, \dots, \pm e_{d-r+1}$ are all protected. Let C_1^+ be the compatibility function given by $C_1^+(1) = 1$ and $C_1^+(i) = *$ for $i \neq 1$, and let C_1^- be the compatibility function given by $C_1^-(1) = -1$ and $C_1^-(i) = *$ for $i \neq 1$. By Lemma 7,

$$|P_{k-1}^{C_1^+}(e_1)| \geq \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}},$$

and the same inequality holds for $|P_{k-1}^{C_1^-}(-e_1)|$. By minimality of $P(S_k)$, there are exactly

$$\sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}$$

protected sites at distance k from the origin. Let C_1^0 be the 1-restriction of C_1^+ . Noting that the sets $P_{k-1}^{C_1^+}(e_1)$, $P_{k-1}^{C_1^-}(-e_1)$ and $P_k^{C_1^0}(0)$ partition the set of protected sites in S_k , we have from the above inequalities and Lemma 4 that

$$|P_k^{C_1^0}(0)| \leq \sum_{i_2=0}^k \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}.$$

It follows that the protected sites in the $(d-2)$ -dimensional sphere $S_k \cap U_0$ embedded in the $(d-1)$ -dimensional space U_0 are minimal. Since this is true for all k satisfying $k_1 \leq k \leq k_1 + c_2$, by induction, it follows that $P(B_{k_1-c_1} \cap U_0)$ is $(d-1, r)$ -canonical set. In particular this means that ie_2 is protected for $i = -k_2, \dots, k_2$, where $k_2 = k_1 - c_1$.

Next we show inductively that the intersection of $P(S_k)$ with the co-dimension 1 sub-space

$$V_i = \{x \in \mathbb{Z}^d : x_2 = i\}$$

is minimal for $i = -k_2, \dots, k_2$. The set $P(S_k \cap V_0)$ is minimal by the same argument we used to prove $P(S_k \cap U_0)$ is minimal. By symmetry, we just have to show that $P(S_k \cap V_i)$ is minimal for $i = 1, \dots, k_2$.

Let C_2^+ be the compatibility function given by $C_2^+(2) = 1$ and $C_2^+(j) = *$ for $j \neq 2$, let C_2^- be the compatibility function given by $C_2^-(2) = -1$ and $C_2^-(j) = *$ for $j \neq 2$, and let C_2^0 be the 2-restriction of C_2^+ . For the induction to go through, we need to add to the hypothesis that

$$|P_{k-i}^{C_2^+}(ie_2)| = \sum_{i_1=0}^{k-i} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}. \quad (20)$$

This does indeed hold for $i = 0$, because

$$|P_k^{C_2^0}(0)| \geq \sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}$$

by Lemma 7, so by minimality of $P(S_k)$ we must have equality here and in (20) in the case $i = 0$. Suppose the claim holds for $i - 1$. Then

$$\begin{aligned} |P_{k-i}^{C_2^0}(ie_2)| &= |P_{k-i}^{C_2^+}(ie_2)| - |P_{k-i-1}^{C_2^+}((i+1)e_2)| \\ &\leq \sum_{i_1=0}^{k-i} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}} - \sum_{i_1=0}^{k-i-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}} \\ &= \sum_{i_2=0}^{k-i} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}. \end{aligned} \quad (21)$$

So $P(S_k \cap V_i)$ is minimal. Furthermore, we must have equality in (21), so (20) holds for $i + 1$. This completes the proof of the claim that $P(S_k \cap V_i)$ is minimal for $i = -k_2, \dots, k_2$.

By induction, $P(B_{k_2} \cap V_i)$ is $(d-1, r)$ -canonical for $i = -k_2, \dots, k_2$. To save space, we shall write P_i for $P(B_{k_2} \cap V_i)$. We have to show two things. First, that each P_i has the same alignment. Second, that each P_i has the same orientation.

We start with the first of these. We shall show that if the set of alignment coordinates of one of the P_i is not $[d-r+1]$ then there are too many protected sites in S_k . Suppose there is a choice of $j \in [d-r+1]$ and $i \in \{-k+1, \dots, k-1\}$ such that j is not a direction of alignment for P_i . We may assume that $d-r+2$ is a direction of alignment for this set instead. Since both e_j and $-e_j$ are protected, Lemma 7 tells us that there are at least

$$2 \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}}$$

protected sites x in S_k with $x_j \neq 0$. By minimality there are exactly

$$\sum_{i_1=0}^k \sum_{i_2=0}^{i_1} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d}{i_{d-r+1}}$$

protected sites in S_k in total, so by Lemma 4 there are at most

$$\sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}} \quad (22)$$

protected sites x in S_k with $x_j = 0$. (In fact we have equality, but we do not need that.) For $-k \leq l \leq k$, let

$$Q_l = \{x \in P_l : x_j = 0\}.$$

If j is an alignment coordinate for P_l then Q_l is $(d-2, r)$ -canonical, while if j is an orientation coordinate for P_l then Q_l is $(d-2, r-1)$ -canonical. In either case,

$$|Q_l \cap S_k| \geq \sum_{i_3=0}^{k-l} \sum_{i_4=0}^{i_3} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-2}{i_{d-r+1}}.$$

Also, $Q_i \cap S_k$ contains at least one more site than this, because by assumption j is an orientation coordinate for P_i , so Q_i is $(d-2, r-1)$ -canonical and therefore

$$|Q_i \cap S_k| \geq \sum_{i_2=0}^{k-i} \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-2}{i_{d-r+1}} \geq \sum_{i_3=0}^{k-i} \sum_{i_4=0}^{i_3} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-2}{i_{d-r+1}} + 1.$$

Hence, by Lemma 4,

$$\sum_{l=-k}^k |Q_l \cap S_k| \geq \sum_{i_2=0}^k \sum_{i_3=0}^{i_2} \cdots \sum_{i_{d-r+1}=0}^{i_{d-r}} \binom{d-1}{i_{d-r+1}} + 1,$$

contradicting (22).

Next we prove the second claim, that each of the $(d-1, r)$ -canonical sets has the same orientation. Note that we could have reversed the roles of e_1 and e_2 in everything we have done so far and obtained $(d-1, r)$ -canonical sets R_{-k}, \dots, R_k in $P(B_k) \cap U_{-k}, \dots, P(B_k) \cap U_k$ respectively. These canonical sets have the same alignment as the canonical sets for the U_k , by the same reasoning as above. The set $(P_{-k} \cup \dots \cup P_k) \cap B_k$ is equal to $P(B_k)$, the set of protected sites in B_k , and so is the set $(R_{-k} \cup \dots \cup R_k) \cap B_k$. Thus

$$(P_{-k} \cup \dots \cup P_k) \cap B_k = (R_{-k} \cup \dots \cup R_k) \cap B_k.$$

It now follows immediately that the orientations of the P_i must match up: if some two are different then we are left with no choice of orientations for R_0 . \square

5. PROOFS OF MAIN THEOREMS

In the previous two sections we proved the three key extremal theorems. Here we use those theorems to derive good approximations to the first and second moments of the number of uninfected sites at time t . This will turn out to be key to proving Theorems 1 and 2.

Let us fix a sequence of probabilities $(p_n)_{n=1}^\infty$, let $E_{d,r}(t, n, x)$ be the event that the site $x \in \mathbb{T}_n^d$ is uninfected at time t , and let $F_{d,r}(t, n, x)$ be the corresponding indicator random variable. We are interested in the total number of uninfected sites at time t , defined to be $F_{d,r}(t, n)$; thus,

$$F_{d,r}(t, n) = \sum_{x \in \mathbb{T}_n^d} F_{d,r}(t, n, x).$$

Often we write $F(t, n)$ for $F_{d,r}(t, n)$. We would like to estimate the first two moments of $F(t, n)$. Together with the Stein-Chen method [28, 13], this will allow us to prove that $F(t, n)$ is asymptotically Poisson distributed, which will enable us to complete the proof of Theorems 1 and 2. The version of Stein-Chen that we shall use is the following formulation due to Barbour and Eagleson [7].

Theorem 13. *Let X_1, \dots, X_n be Bernoulli random variables with $\mathbb{P}(X_i = 1) = p_i$. Let $Y_n = \sum_{i=1}^n X_i$, and let $\lambda_n = \mathbb{E}(Y_n) = \sum_{i=1}^n p_i$. For each $i \in [n]$, let $N_i \subset [n]$ be such that X_i is independent of $\{X_j : j \notin N_i\}$. For each $i, j \in [n]$, let $p_{ij} = \mathbb{P}(X_i X_j = 1)$. Let $Z_n \sim \text{Po}(\lambda_n)$. Then*

$$\sup_{A \subset \mathbb{Z}} |\mathbb{P}(Y_n \in A) - \mathbb{P}(Z_n \in A)| \leq \min\{1, \lambda_n^{-1}\} \left(\sum_{i=1}^n \sum_{j \in N_i} p_i p_j + \sum_{i=1}^n \sum_{j \in N_i \setminus \{i\}} p_{ij} \right).$$

The expectation of $F(t, n)$ is $\mathbb{E}F(t, n) := \lambda(t, n) = n^d \rho_1$, where

$$\rho_1 := \mathbb{P}_{p_n}(E_{d,r}(t, n, x)).$$

We also need to bound the quantity

$$\rho_2 := \max\{\mathbb{P}_{p_n}(E_{d,r}(t, n, x) \cap E_{d,r}(t, n, y)) : \|x - y\| \leq 2t\},$$

which we shall use to bound the p_{ij} in Theorem 13. The condition $\|x - y\| \leq 2t$ is equivalent to the statement that the events $E_{d,r}(t, n, x)$ and $E_{d,r}(t, n, y)$ are dependent.

The following lemma is a simple computation and is similar to Lemma 18 of [9].

Lemma 14. Let $t = o((\log n / \log \log n)^{d-r+1})$ and let

$$q_n = O(n^{-d/m_{d,r}(t)} \log n). \quad (23)$$

Then for any constant $c > 0$ we have $t^c q_n = o(1)$. \square

Lemma 15. Let $t = o((\log n / \log \log n)^{d-r+1})$ and let q_n satisfy (23). Then

$$\rho_1 = (1 + o(1)) g_{d,r} q_n^{m_{d,r}(t)}, \quad (24)$$

where

$$g_{d,r} = \binom{d}{d-r+1} 2^{r-1} d^{2(d-r+1)}.$$

Furthermore,

$$\rho_2 = O(q_n \rho_1) = o(\rho_1). \quad (25)$$

Proof. We only sketch the proof, since it is similar to the proofs of several lemmas in Section 4 of [9]. Let $g_{d,r}(t, k)$ be the number of arrangements of $m_{d,r}(t) + k$ uninfected sites in B_t such that the origin is protected. Thus,

$$\rho_1 = \sum_{k=0}^{|B_t| - m_{d,r}(t)} g_{d,r}(t, k) p_n^{|B_t| - m_{d,r}(t) - k} q_n^{m_{d,r}(t) + k}. \quad (26)$$

Theorem 11 implies that

$$g_{d,r}(t, 0) = \binom{d}{d-r+1} 2^{r-1} d^{2(d-r+1)} = g_{d,r},$$

while Theorem 12 allows us to bound $g_{d,r}(t, k)$ for general k by

$$g_{d,r}(t, k) = O(t^{O(k)}). \quad (27)$$

It is now easy to see that these last two equations combined with (26) give the bound we want on ρ_1 in (24).

For (25), we follow Lemmas 18 and 19 of [9]. The key point is that a set of sites cannot be a semi-canonical set for two distinct sites, so if x and y are protected then $B_t(x) \cup B_t(y)$ contains at least $m_{d,r}(t) + 1$ uninfected sites. This means that

$$\rho_2 \leq \sum_{k=0}^{2|B_t|} h_{d,r}(t, k) q_n^{m(t)+1+k},$$

where $h_{d,r}(t, k)$ is the number of configurations of $m(t) + 1 + k$ uninfected sites in $B_t(x) \cup B_t(y)$ such that both x and y are protected. Using the bound on $g_{d,r}(t, k)$ from (27) we obtain a similar bound on $h_{d,r}(t, k)$, namely

$$h_{d,r}(t, k) = O(t^{O(k)}).$$

We are now able to estimate ρ_2 in much the same way that we estimated ρ_1 . \square

The following is an easy consequence of Lemma 15 and Theorem 13.

Theorem 16. Let $2 \leq r \leq d$, let $t = o((\log n / \log \log n)^{d-r+1})$, let q_n satisfy (23), and let $X(t, n) \sim \text{Po}(\lambda(t, n))$. Then

$$\sup_{A \subset \mathbb{Z}} |\mathbb{P}(F(t, n) \in A) - \mathbb{P}(X(t, n) \in A)| = O(t^d q_n) = o(1). \quad \square$$

Proof of Theorem 1. The probability of percolating in time at most t is increasing in p , by a standard coupling argument. Therefore, we may assume that the usual bound (23) on q_n holds. Theorem 16 tells us that $F(t, n)$ converges in distribution to $\text{Po}(\lambda(t, n))$, so

$$\mathbb{P}(T \leq t) = \mathbb{P}_{p_n}(F(t, n) = 0) = (1 + o(1))e^{-\lambda(t, n)}.$$

We have

$$\lambda(t, n) = n^d \rho_1 = n^d \Theta(1) q_n^{m_{d,r}(t)}$$

by Lemma 15. Therefore,

$$\mathbb{P}_{p_n}(T \leq t) \rightarrow \begin{cases} 1 & \text{if } q_n \leq (n^{-d}/\omega(n))^{1/m_{d,r}(t)}, \\ 0 & \text{if } q_n \geq (n^{-d}\omega(n))^{1/m_{d,r}(t)}, \end{cases}$$

for some function $\omega(n) \rightarrow \infty$, as required. \square

Proof of Theorem 2. Part (i) of the theorem is an immediate corollary of Theorem 1. For part (ii), we are given that q_n satisfies

$$(n^{-d}/\omega(n))^{1/m_{d,r}(t)} \leq q_n \leq (n^{-d}\omega(n))^{1/m_{d,r}(t)}$$

for all $\omega(n) \rightarrow \infty$. Observe that if

$$\omega(n) = \exp\left(c_0 \frac{\log n}{t}\right)$$

for a sufficiently small constant c_0 , then

$$(n^{-d}/\omega(n))^{1/m_{d,r}(t)} \geq (n^{-d}\omega(n))^{1/m_{d,r}(t-1)}.$$

Therefore, $\mathbb{P}_{p_n}(T \leq t-1) = o(1)$ by Theorem 1. Similarly we have $\mathbb{P}_{p_n}(T \geq t+2) = o(1)$. So $T \in \{t, t+1\}$ with high probability.

Now let $q_n^{m_{d,r}(t)} n^d \rightarrow c$ as $n \rightarrow \infty$. Then

$$\mathbb{P}_{p_n}(T = t) \sim \mathbb{P}_{p_n}(T \leq t) \sim e^{-\lambda(t, n)} \sim \exp(-n^d g_{d,r} q_n^{m_{d,r}(t)}) \sim \exp(-g_{d,r} c).$$

Since $T \in \{t, t+1\}$ with high probability, we must also have

$$\mathbb{P}_{p_n}(T = t+1) \sim 1 - \exp(-g_{d,r} c). \quad \square$$

That completes the proofs of the main results. We now briefly turn our attention to the rather easier setting of subcritical models.

6. SUBCRITICAL MODELS

In this final section we give a complete description of the time for percolation in the case of subcritical models. Here, subcritical means that there exist closed cofinite sets; in d dimensions, this means that the threshold r is strictly greater than d . Our description of the percolation time is valid for all p for which percolation occurs with high probability; we determine how large $q = 1 - p$ needs to be for this to be the case, and as a corollary we determine the critical probabilities for percolation under subcritical models on the torus.

Lemma 17. *Let $3 \leq d+1 \leq r \leq 2d$ and $k \geq 0$. Suppose the origin is protected under the r -neighbour model. Then*

$$|P(B_k)| \geq \binom{2d-r+1}{k}. \quad (28)$$

Proof. The proof is the usual inductive double counting argument. Write P_k for $P(S_k)$ and consider the bipartite graph H_k with vertex sets P_{k-1} and P_k , and edges induced by \mathbb{Z}^d . Note that $|P_0| = 1$, as claimed. If $x \in P_k$ then x has support at most k , so the degree of x in H_k is at most k . If $y \in P_{k-1}$ then, since y is protected under the r -neighbour model, it follows that y can have at most $r-1$ infected neighbours at time $(t-k-1)$, and hence it has at least $2d-r+1$ uninfected neighbours. Of these, at most $k-1$ are in P_{k-2} , since y has support at most $k-1$, so y has at least $2d-r-k+2$ protected neighbours in P_k .

We have proved that

$$j|P_k| \geq (2d-r-k+2)|P_{k-1}|,$$

which implies (28). \square

The above proof holds for any r , but it is only tight for $r \geq d+1$.

For $d > r$, a subset K of $B_t(x)$ is (d, r) -canonical if there is a subset I of $[d]$ of size $2d-r+1$ and $\epsilon_i \in \{-1, 1\}$ for each $i \in I$ such that

$$K = \{y \in B_t(x) : y_i - x_i = \epsilon_i \text{ for all } i \in I \text{ and } y_i = x_i \text{ otherwise}\}.$$

Lemma 18. *Let $3 \leq d+1 \leq r \leq 2d$ and $t \geq 0$. Suppose the origin is protected and $P(B_t)$ is minimal. Then $P(B_t)$ is (d, r) -canonical.*

We omit the proof since it is similar to the proof of Theorem 11.

Theorem 19. *Let $3 \leq d+1 \leq r \leq 2d$ and $t \geq 0$. Let $(p_n)_{n=1}^\infty$ be a sequence of probabilities and let $\omega(n) \rightarrow \infty$.*

(i) *If $t \leq 2d-r+1$ and*

$$n^{-d/m_{d,r}(t-1)}\omega(n) \leq q_n \leq n^{-d/m_{d,r}(t)}/\omega(n),$$

then $T = t$ with high probability.

(ii) *If $t \leq 2d-r+1$ and $q_n^{m_{d,r}(t)} n^d \rightarrow c$ as $n \rightarrow \infty$, for some constant $c > 0$, then $T \in \{t, t+1\}$ with high probability.*

(iii) *If*

$$q_n \geq n^{-d/m_{d,r}(2d-r+1)}\omega(n) = n^{-d/2^{2d-r+1}}\omega(n), \quad (29)$$

then $T = \infty$ with high probability.

No stability theorem is needed for the proof of Theorem 19 because if percolation occurs then the assertion is that it occurs in a time that does not depend on n , so we only ever need to consider configurations inside ℓ_1 balls of bounded size.

Proof. Parts (i) and (ii) of the theorem follow using the same methods as Theorem 1, so we only have to prove (iii). Let $m = 2d-r+1$. It is sufficient to show that if q_n satisfies (29) then with high probability \mathbb{T}_n^d contains an uninfected m -dimensional hypercube at time 0. More specifically, if all the sites in some translate of

$$\{0, 1\}^m \times \{0\}^{d-m}$$

are initially uninfected, then they remain uninfected forever, so it suffices to prove that there exists an empty such translate with high probability. There are $n^{d2^{-m}}$ disjoint hypercubes of the given form, and the probability that none is initially uninfected is

$$(1 - q_n^{2^m})^{n^{d2^{-m}}} \leq \exp(-q_n^{2^m} n^{d2^{-m}}) \leq \exp(-2^{-m}\omega(n)^{2^m}),$$

which is $o(1)$. \square

Finally, we note the corollary mentioned at the beginning of the section, which concerns the critical probabilities of subcritical models. Let $P(d, r, n, p)$ be the probability that a random set $A \subset \mathbb{T}_n^d$ percolates under the r -neighbour process, where sites are included in A independently with probability p .

Corollary 20. *Let $3 \leq d + 1 \leq r \leq 2d$ and let $(p_n)_{n=1}^\infty$ be a sequence of probabilities.*

(i) *If $q_n = O(n^{-d/m_{d,r}(2d-r+1)})$, then $P(d, r, n, p_n) \rightarrow 1$ as $n \rightarrow \infty$.*

(ii) *If $q_n \gg n^{-d/m_{d,r}(2d-r+1)}$, then $P(d, r, n, p_n) \rightarrow 0$ as $n \rightarrow \infty$.* □

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